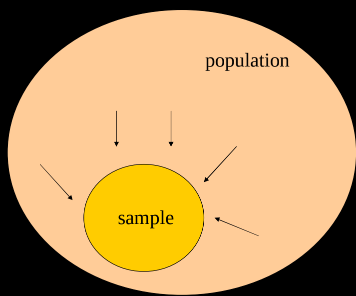
Probability —— deductive reasoning

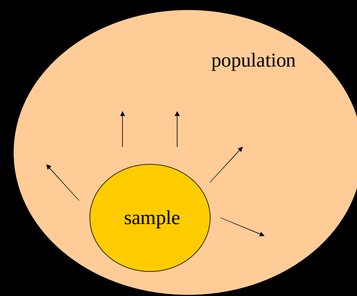


$\Pr(H) = \Pr(T) = 0.5$

ten times

$\Pr(\text{all heads}) = (0.5)^{10}$

Statistical Inference —— inductive reasoning



$\Pr(H) = \Pr(T) = ?$

ten times

All heads.

Is it a fair coin?

Beginning of Probability

C. de Mére → ?

Pascal

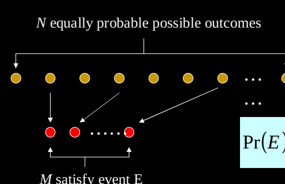
Fermat

Head → A get 1 point
Tail → B get 1 point
4 points → win $100
Interrupted at 3:2

$100 →(?) A $75
$100 →(?) B $25

$\Pr(\text{B win}) = \Pr(\text{both tails}) = 0.25$

Classical School



$\Pr(E) = \frac{M}{N}$

$\Pr(\text{heart}) = \frac{13}{52} = 0.25$

$\Pr(\text{odd number from dice toss}) = \frac{3}{6} = 0.5$

$\Pr(\text{different birthdays}) = \frac{365\times 364\times\cdots\times(365-n+1)}{365\times 365\times\cdots\times 365} = \frac{_{365}P_n}{(365)^n}$

Frequency School

Results of coin toss experiment

No. of Tosses	5	100	1000	1000000
No. of Heads	3	47	508	499987
Relative Frequency of Heads	0.6	0.47	0.508	0.499987

$\text{relative frequency of Heads} \longrightarrow 0.5 \quad \text{as} \quad n \to \infty$

$$\Pr(E) \overset{define}{=} \lim_{n\to\infty} \frac{f}{n}$$

Subjective School

Probabilities are quantitative expressions of uncertain about a person's knowledge of the occurrence of some event.

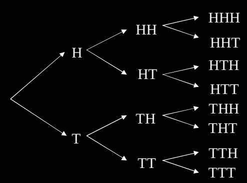
"I thought there is 30% chance that tomorrow will rain."

"Arsenal will have more than 80% chance to win the championship this year."

"I wished I had worked hard in last semester. I think I should have 80% chance to get better grade if I did so."

Counting Procedures

Example: Possible outcomes of tossing three coins



$N = 2 \times 2 \times 2 = 8$

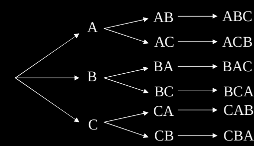
n_1 ways in 1st step n_2 ways in 2nd step n_k ways in kth step

$n_1 n_2 \cdots n_k$ ways in k steps

Permutation

A specific arrangement of objects in a definite order.

Example: possible arrangements of three persons in a row



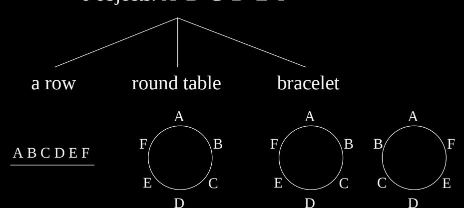
$N = 3 \times 2 \times 1 = 6$

No. of permutations of n distinct objects

$$= n(n-1)\cdots(2)(1) = n!$$

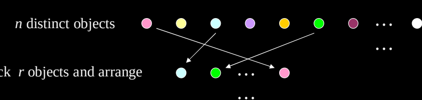
Permutation

6 objects: A B C D E F



6! = 720 (6-1)! = 120 (6-1)!/2 = 60

Permutation



No. of ordered subset:

$${}_nP_r = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

Example: Tierce outcomes in horse race with 14 horses

5 6 14 7 1 2 1 2 7 …

$N = {}_{14}P_3 = \frac{14!}{11!} = 2184$

Combination

A subset of *r* objects from *n* distinct objects, without regard to order.

Example: Choose three letters from {A, B, C, D}

Combination	Permutations
{A, B, C}	(A,B,C),(A,C,B),(B,A,C),(B,C,A),(C,A,B),(C,B,A)
{A, B, D}	(A,B,D),(A,D,B),(D,A,B),(D,B,A),(B,A,D),(B,D,A)
{A, C, D}	(A,C,D),(A,D,C),(C,A,D),(C,D,A),(D,A,C),(D,C,A)
{B, C, D}	(B,C,D),(B,D,C),(C,B,D),(C,D,B),(D,B,C),(D,C,B)

$${}_4C_3 = \frac{{}_4P_3}{3!} = \frac{24}{6} = 4 \qquad {}_4P_3 = \frac{4!}{(4-3)!} = \frac{24}{1} = 24$$

$${}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!}$$

Combination

Example: 6 numbers chosen from {1, 2, ….., 49}

$N = {}_{49}C_6 = \frac{49!}{6!\,43!} = 13983816$

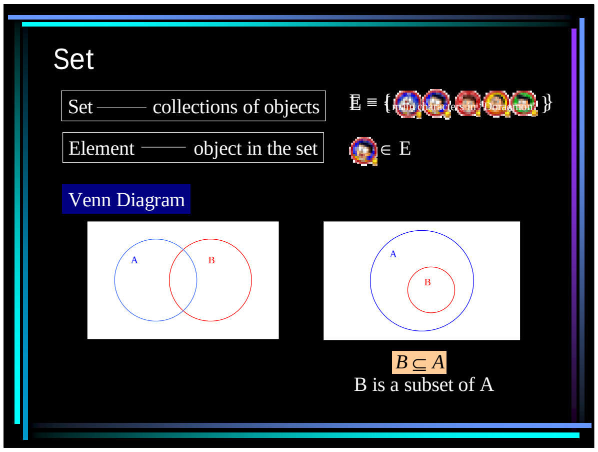
Example: draw five poker cards, Pr(full house) = ?

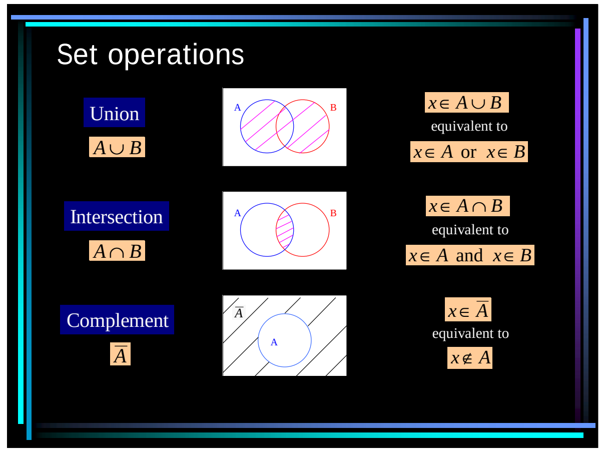
Full house: 

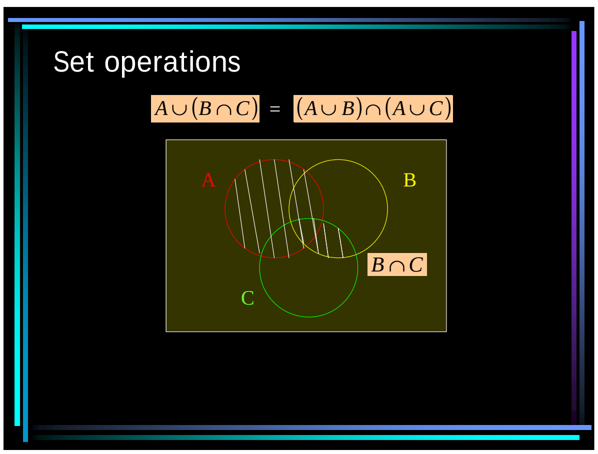
no. of possible outcomes: $N = {}_{52}C_5 = 2598960$

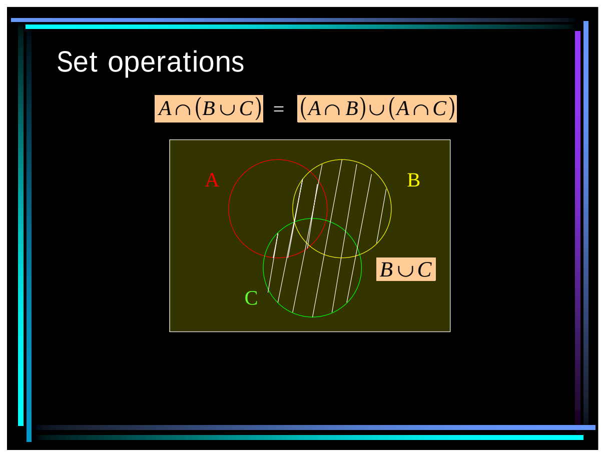
no. of possible full houses: $M = {}_{13}C_2 \times 2 \times {}_4C_3 \times {}_4C_2 = 3744$

$$\Pr(\text{full house}) = \frac{3744}{2598960} = 0.00144$$


Set
Set —— collections of objects
Element —— object in the set
E = { }
∈ E
Venn Diagram
A
B
A
B
$B \subseteq A$
B is a subset of A


Set operations
Union
$A \cup B$
$x \in A \cup B$
equivalent to
$x \in A$ or $x \in B$
Intersection
$A \cap B$
$x \in A \cap B$
equivalent to
$x \in A$ and $x \in B$
Complement
$\overline{A}$
$x \in \overline{A}$
equivalent to
$x \notin A$


Set operations
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
A
B
C
$B \cap C$


Set operations
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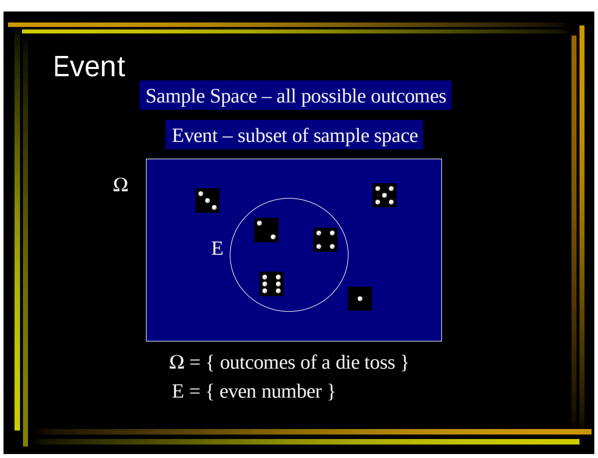
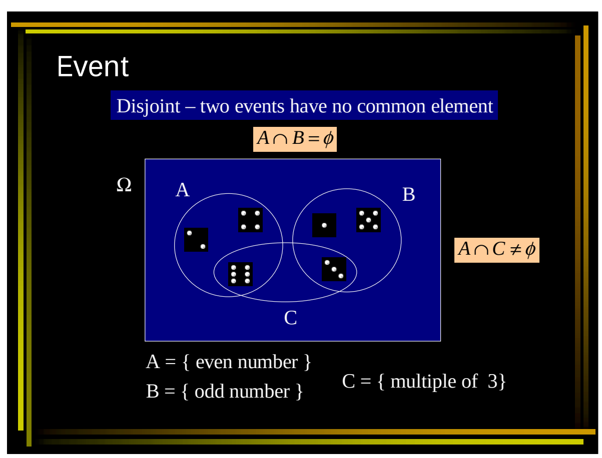
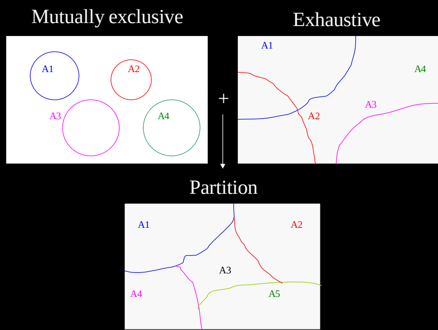
A
B
C
$B \cup C$

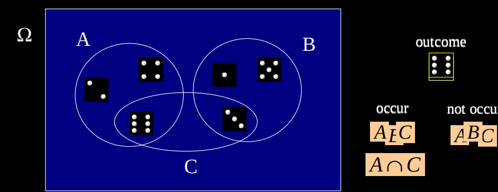

Event
Sample Space – all possible outcomes
Event – subset of sample space
Ω
E
Ω = { outcomes of a die toss }
E = { even number }


Event
Disjoint – two events have no common element
$A \cap B = \phi$
Ω
A
B
C
$A \cap C \neq \phi$
A = { even number }
B = { odd number }
C = { multiple of 3}

Event



Occurrence — E occurs if the outcome belongs to E.



A = { even number }
B = { odd number }
C = { multiple of 3}

Komogorov's Axiom System

$\Pr(E) \geq 0$ for any event E $\quad \Pr(\Omega) = 1$

$\Pr(A \cup B) = \Pr(A) + \Pr(B)$ if $A \cap B = \phi$

$\Pr(\phi) = 0$

If $A \subseteq B$, then $\Pr(A) \leq \Pr(B)$

$\Pr(\overline{A}) = 1 - \Pr(A)$

$\Pr(E) \leq 1$ for any event E

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

Example

$\Pr(M) = 0.1$

$\Pr(A) = 0.2$

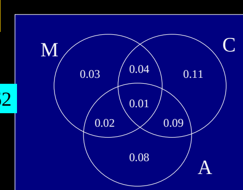
$\Pr(C) = 0.25$

$\Pr(M \cap C) = 0.05$

$\Pr(M \cap A) = 0.03$

$\Pr(C \cap A) = 0.1$

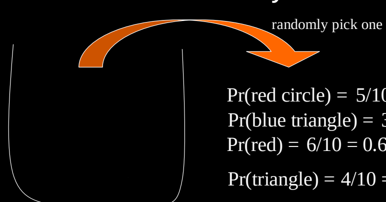
$\Pr(M \cap C \cap A) = 0.01$



Pr(none of these behaviors) = 0.62

$\Pr(M \cup C \cup A) = 0.38$

Conditional Probability



Pr(red circle) = 5/10 = 0.5
Pr(blue triangle) = 3/10 = 0.3
Pr(red) = 6/10 = 0.6

Pr(triangle) = 4/10 = 0.4

You are told that it is blue.

Pr(triangle) = 3/4 = 0.75

$\Pr(\text{triangle} \mid \text{blue}) = \frac{3/10}{4/10} = \frac{\Pr(\text{blue triangle})}{\Pr(\text{blue})}$

You are told that it is red.

Pr(triangle) = 1/6 = 0.17

$\Pr(\text{triangle} \mid \text{red}) = \frac{1/10}{6/10} = \frac{\Pr(\text{red triangle})}{\Pr(\text{red})}$

Conditional Probability

For any two events A and B, the conditional probability of A given that B has occurred is written as Pr(A|B) and defined as

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Multiplication theorem

$$\Pr(A \cap B) = \Pr(B)\Pr(A \mid B)$$

$$\Pr(A \cap B \cap C) = \Pr(C)\Pr(B \mid C)\Pr(A \mid B \cap C)$$

Independence

Two events A and B are called *independent* if and only if

$$\Pr(A \cap B) = \Pr(A)\Pr(B)$$

$$\Rightarrow \Pr(B) = \Pr(B \mid A) \text{ if } \Pr(A) > 0$$

Example : opinion poll on abortion

	Favor (F)	Opposed (O)
White (W)	0.459	0.441
Non-white (N)	0.051	0.049

$\Pr(W) = 0.459 + 0.441 = 0.9$ $\quad \Pr(F) = 0.459 + 0.051 = 0.51$

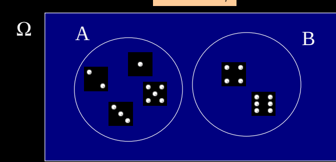
$\Pr(W \cap F) = 0.459 \quad = \quad \Pr(W)\Pr(F) = 0.459$

W and F are independent.

Independence vs Mutually Exclusive

Disjoint – two events have no common element

$$A \cap B = \phi$$



A and B are not independent !

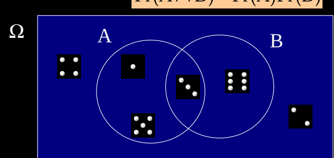
$$\Pr(A \cap B) = 0$$

$$\neq \Pr(A)\Pr(B) \quad \text{if } \Pr(A) > 0, \Pr(B) > 0$$

Independence vs Mutually Exclusive

Independence – events have no addition information to each other

$$\Pr(A \cap B) = \Pr(A)\Pr(B)$$



A = { odd }

B = {multi of 3}

$$A \cap B = \{3\} \neq \phi$$

A and B are independent ! '

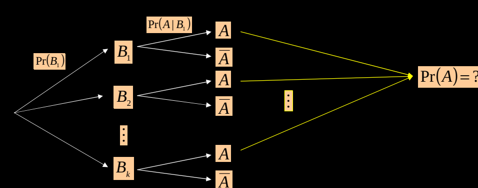
$$\Pr(A \cap B) = 1/6 = 1/2 \times 1/3 = \Pr(A)\Pr(B)$$

Law of Total Probability

$B_1, B_2, \ldots, B_k$ —— partition of the sample space

mutually exclusion —— cannot occur at the same time

exhaustive —— at least one of them will occur



$$\Pr(A) = \Pr(B_1)\Pr(A \mid B_1) + \cdots + \Pr(B_k)\Pr(A \mid B_k)$$

Law of Total Probability

Example: Morse code transmission

Bayes

dot : dash = 3 : 4

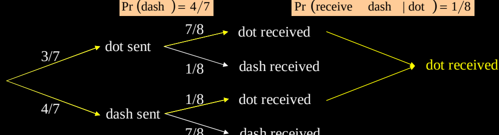
$\Pr(\text{dot}) = 3/7$

$\Pr(\text{dash}) = 4/7$

error rate = 1/8

$\Pr(\text{receive dot} \mid \text{dash}) = 1/8$

$\Pr(\text{receive dash} \mid \text{dot}) = 1/8$



$$\Pr(\text{receive dot}) = \frac{3}{7} \times \frac{7}{8} + \frac{4}{7} \times \frac{1}{8} = \frac{25}{56}$$

$$\Pr(\text{dot sent} \mid \text{dot received}) = ?$$

Bayes Rule

$B_1, B_2, \ldots, B_k$ —— partition of the sample space

$$\Pr(B_i \mid A) = \frac{\Pr(B_i \cap A)}{\Pr(A)} = \frac{\Pr(B_i)\Pr(A \mid B_i)}{\Pr(A)}$$

$$= \frac{\Pr(B_i)\Pr(A \mid B_i)}{\Pr(B_1)\Pr(A \mid B_1) + \Pr(B_2)\Pr(A \mid B_2) + \cdots + \Pr(B_k)\Pr(A \mid B_k)}$$

Example: Morse code transmission

$$\Pr(\text{dot sent} \mid \text{dot received}) = \frac{\Pr(\text{dot sent})\Pr(\text{dot received} \mid \text{dot sent})}{\Pr(\text{dot received})}$$

$$= \frac{3/7 \times 7/8}{25/56} = 0.84$$

Bayes Rule

Example: virus test

A —— Test shows positive result as carrier

B —— Infected by the virus

0.3% infected → $\Pr(B) = 0.003$ ← prior

2% false positive → $\Pr(A \mid \bar{B}) = 0.02$

1% false negative → $\Pr(\bar{A} \mid B) = 0.01$

$$\Pr(B \mid A) = \frac{\Pr(B)\Pr(A \mid B)}{\Pr(B)\Pr(A \mid B) + \Pr(\bar{B})\Pr(A \mid \bar{B})}$$

$$= \frac{0.003 \times 0.99}{0.003 \times 0.99 + 0.997 \times 0.02} = 0.1296$$

↑ posterior